# Extending INLA to a class of near-Gaussian latent models

Thiago G. Martins

**Joint work with Håvard Rue**
Norwegian University of Science and Technology
Trondheim, Norway

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# Latent Gaussian Models (LGMs)

- Hierarchical representation

**Stage 1.** $\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}_1 \sim \pi(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}_1) = \prod_{i=1}^{n_d} \pi(y_i|x_i, \boldsymbol{\theta}_1)$.

**Stage 2.** $\mathbf{x}|\boldsymbol{\theta}_2 \sim \pi(\mathbf{x}|\boldsymbol{\theta}_2) = \mathcal{N}(\mathbf{x}; \mathbf{0}, \mathbf{Q}^{-1}(\boldsymbol{\theta}_2))$,

**Stage 3.** $\boldsymbol{\theta} \sim \pi(\boldsymbol{\theta})$.

- Our aim

**Stage 2$^{\text{new}}$.** $\underbrace{(\mathbf{x}_G, \mathbf{x}_{NG})}_{\mathbf{x}}|\theta_2 \sim \pi(\mathbf{x}|\theta_2)$

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- $\pi(\mathbf{x}_{NGi}|\boldsymbol{\theta}_2)$ not very far from Gaussian.

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1. For some specific reason, people need such models

   - E.g. gamma frailty models in survival analysis literature - log-gamma random effects.

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# INLA - main objective

Posterior

$$\pi(\mathbf{x}, \boldsymbol{\theta}|\mathbf{y}) \propto \pi(\boldsymbol{\theta})\, \pi(\mathbf{x}|\boldsymbol{\theta}) \prod_{i \in \mathcal{I}} \pi(y_i|x_i, \boldsymbol{\theta})$$

Compute the posterior marginals:

$$\pi(x_i|\mathbf{y}) = \int \pi(\boldsymbol{\theta}|\mathbf{y})\, \pi(x_i|\boldsymbol{\theta}, \mathbf{y})\, d\boldsymbol{\theta}$$

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$$\pi(\mathbf{x}|\mathbf{y}) \propto \exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \sum_i \log \pi(y_i|x_i)\right)$$

Constructed as follows:

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- Computational point of view

  - Conditional independence properties in the Gaussian latent field translate into a sparse precision matrix which speed up the GMRF approximation considerably.

# The extension

1. Approximate $\pi(\mathbf{x}_{NG}|\boldsymbol{\theta}_2)$ by a Gaussian $\pi_G(\mathbf{x}_{NG}|\boldsymbol{\theta}_2)$
2. Correct for this approximation in the likelihood function with a correction term

$$CT_i = \pi(\mathbf{x}_{NG,i}|\boldsymbol{\theta}_2)/\pi_G(\mathbf{x}_{NG,i}|\boldsymbol{\theta}_2), \;\; i = 1, ..., k$$

- **But why?**

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# The extension (cont.)

- Affect mainly the Gaussian approximation to $\pi(\mathbf{x}|\mathbf{y}, \boldsymbol{\theta})$

**Before:**

$$\pi(\mathbf{x}|\mathbf{y}) \propto \exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \sum_i \log \pi(y_i|x_i)\right)$$

**Now:**

$$\pi(\mathbf{x}|\mathbf{y}) \propto \exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \sum_{i=1}^{n_d} \log \pi(y_i|x_i) + \sum_{j=1}^{k} h_j(x_{NG,j})\right)$$

where $h_j(x_{NG,j}) = \log[\pi(\mathbf{x}_{NG,j})/\pi_G(x_{NG,j})]$.

- A good Gaussian approximation will depend heavily on the shape of $h_j(x_{NG,j})$.

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# Correction term

- It is desirable to have a bounded correction term

$$CT = \pi(\mathbf{x}_{NG}|\boldsymbol{\theta}_2)/\pi_G(\mathbf{x}_{NG}|\boldsymbol{\theta}_2) < \infty$$

  at least on the region that concentrates the bulk of probability mass.
- Although not necessary, it would be nice to have a log-concave correction term, at least around the mode of $\pi(\mathbf{x}|\mathbf{y}, \boldsymbol{\theta})$.
  - More stability to the optimization step.
- In our examples, we have chosen $\pi_G(x_{NG,j}|\theta)$ to be a Gaussian with zero mean and low precision $\tau \to 0$.

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# Survival analysis with gamma frailty

1. $y_{i,j} \sim \exp(\lambda_{i,j}), \quad i = 1, ..., I, \ j = 1, ..., J$

2. $\eta_{i,j} = \log(\lambda_{i,j}) = \beta_0 + \beta_1 x_{i,j} + \log w_i$
   $w_i \sim gamma(\kappa, \kappa), \ E(w_i) = 1$
   $\beta_i \sim N(0, 10^2)$

3. $\kappa \sim gamma(1, 0.1)$

- For this model $\mathbf{x} = (\boldsymbol{\eta}, \boldsymbol{\beta}, \mathbf{b})$, with $\mathbf{b} = \log(\mathbf{w})$.
- Cannot be applied directly with INLA.

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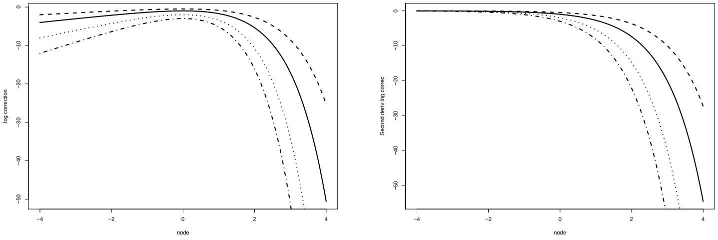
# Survival analysis with gamma frailty (cont.)

- Apply our approach with a low precision Gaussian.
- In this case we get a log-concave correction term.




(a) (b)

Figura: Plot of the log correction term (a) and of the second derivative of the log correction term (b) against $b_i$ for different values of $\kappa$

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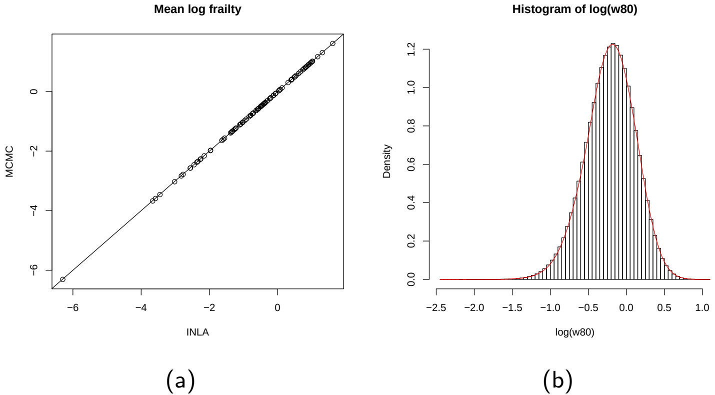


Figura: (a) Plot of the posterior mean of the log frailties returned by INLA (x-axis) vs. MCMC (y-axis). (b) Approximate posterior density for $\log w_{80}$ obtained by INLA (solid line) and by MCMC (histogram).

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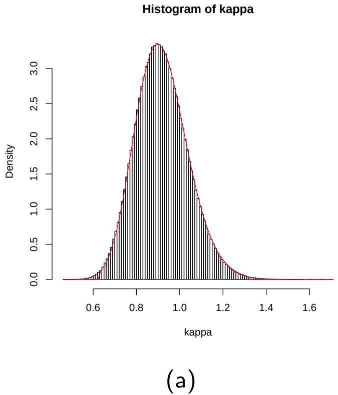


Figura: (a) Approximate posterior density for $\kappa$ obtained by INLA (solid line) and by MCMC (histogram) (b) Approximate posterior density for $\beta_1$ obtained by INLA (solid line) and by MCMC (histogram)

# Robust mixed-effects model

$$\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + b_i + \mathbf{e}_i, \quad \dim(\mathbf{y}_i) = k_i$$

$$b_i \sim N(0, \psi_b^2), \quad \mathbf{e}_i \sim N(0, \psi_e^2\mathbf{I}),$$

- Gaussian distributions vulnerable to outliers
- Provides outlier accommodation together with outlier identification.

1. $y_{ij}|\mathbf{x}, \boldsymbol{\theta} \sim t(\eta_{ij}, \psi_e^2, \nu), \quad i = 1, ..., n$ and $j = 1, ..., k_i$,
2. $\eta_{ij} = \mathbf{X}_i\beta + b_i$
3. $\psi_b^2, \psi_e^2 \sim gamma(1, 0.1)$, $\nu^* = \log(\nu - 5) \sim N(3, 1)$

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$$\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + b_i + \mathbf{e}_i, \quad \dim(\mathbf{y}_i) = k_i$$

$$b_i \sim t(0, \psi_b^2, \nu), \quad \mathbf{e}_i \sim t(0, \psi_e^2\mathbf{I}, \nu),$$

- Gaussian distributions vulnerable to outliers
- Provides outlier accommodation together with outlier identification.

1. $y_{ij}|\mathbf{x}, \boldsymbol{\theta} \sim t(\eta_{ij}, \psi_e^2, \nu), \quad i = 1, ..., n$ and $j = 1, ..., k_i$,
2. $\eta_{ij} = \mathbf{X}_i\beta + b_i$
3. $\psi_b^2, \psi_e^2 \sim gamma(1, 0.1)$, $\nu^* = \log(\nu - 5) \sim N(3, 1)$

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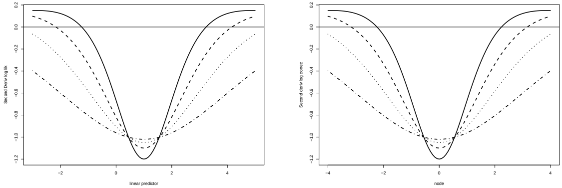
# Robust mixed-effects model (cont.)

- Again, if we use a zero mean and low precision Gaussian ($\mu = 0$, $\tau \to 0$)

$$\log CT_i = -\frac{(\nu + 1)}{2} \log \left\{ 1 + \frac{b_i^2}{\psi_b^2 \nu} \right\} + \text{const.}$$

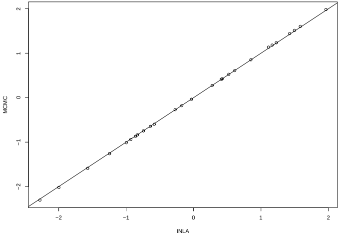



(a) (b)

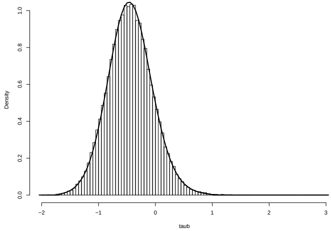
Figura: (a) Plot of the second derivative of the log likelihood against linear predictor and (b) plot of the second derivative of the log correction term for different values of $\nu$.

# Robust mixed-effects model (cont.)

- Again, similar accuracy with only a fraction of computational time.

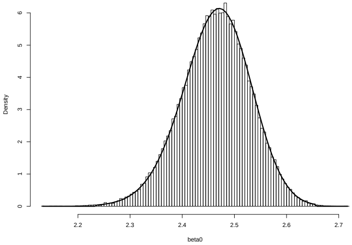



(a) log(**b**) returned by INLA (x-axis) vs. MCMC (y-axis).

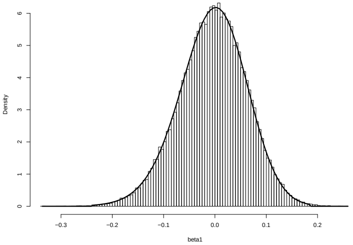



(b) $\log \tau_b = \log 1/\psi_b^2$.

# Robust mixed-effects model (cont.)




(c) $\log \beta_0$.




(d) $\log \beta_1$.

# Conclusion

- Extends the INLA method to latent models, where independent components of the latent field can have a *near-Gaussian* distribution.
- For the previous examples, when compared with MCMC, our method shows similar accuracy with only a small fraction of computation time.
- No precise definition of *near-Gaussian* distribution is given, the main interest is on how our approximation works in practice.

# Future

- Which class of distributions can be used in this framework.
- Other useful classes of models deserve further investigation. E.g. dynamic models with non-Gaussian error term for the observation and/or system equation, as in [Kitagawa, 1987].
- Develop a systematic approach to model selection using this framework to test standard assumptions of a given model, e.g. Gaussian random effects.

THANK YOU

Kitagawa, G. (1987).
Non-Gaussian state-space modeling of nonstationary time series.
*Journal of the American statistical association*, 82(400):1032–1041.

Pinheiro, J., Liu, C., and Wu, Y. (2001).
Efficient algorithms for robust estimation in linear mixed-effects models using the multivariate t distribution.
*Journal of Computational and Graphical Statistics*, 10(2):249–276.