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An infinitesimal sensitivity measure for Bayesian hierarchical models

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Roos and Held (2011):

– New sensitivity measure S based on the Hellinger distance with respect to changes in the prior distribution

– The whole amount of information contained in the prior densities and marginal posterior densities obtained from `inla` (Rue, Martino, and Chopin 2009) is used

– Calibration with respect to the Bernoulli distribution

Hellinger Distance

Le Cam (1986)

$$H(f_0, f_1) = \sqrt{\frac{1}{2}\int_{-\infty}^{\infty}\left(\sqrt{f_0(u)} - \sqrt{f_1(u)}\right)^2 du} = \sqrt{1 - BC(f_0, f_1)},$$

where

$$BC(f_0, f_1) = \int_{-\infty}^{\infty} \sqrt{f_0(u) f_1(u)} du$$

Hellinger Distance

Bernoulli:

$$H(\mathrm{Ber}(\pi_0), \mathrm{Ber}(\pi_1)) = \sqrt{1 - \sqrt{(1-\pi_0)(1-\pi_1)} - \sqrt{\pi_0 \pi_1}}$$

Normal:

$$H\Big(\mathrm{N}(\mu_0, \sigma_0^2), \mathrm{N}(\mu_1, \sigma_1^2)\Big) = \sqrt{1 - \sqrt{\frac{2\sigma_0\sigma_1}{\sigma_0^2+\sigma_1^2}} \exp\Big(-\frac{(\mu_0-\mu_1)^2}{4(\sigma_0^2+\sigma_1^2)}\Big)}$$

Gamma:

$$H\Big(\mathrm{G}(\alpha_0, \beta_0), \mathrm{G}(\alpha_1, \beta_1)\Big) = \sqrt{1 - \Gamma\Big(\frac{\alpha_0+\alpha_1}{2}\Big)\sqrt{\frac{\beta_0^{\alpha_0}\beta_1^{\alpha_1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)(\frac{\beta_0+\beta_1}{2})^{\alpha_0+\alpha_1}}}}$$

Hellinger Distance

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Calibration of the Hellinger distance with respect to the Normal distribution

$H(\mathrm{N}(0, 1), \mathrm{N}(\mu(h), 1)) = h$

$h = h(\mu)$	$\mu(h) = \mu$	$h = h(\mu)$	$\mu(h) = \mu$
0.000	0	0.0	0.000
0.343	1	0.1	0.284
0.627	2	0.2	0.571
0.822	3	0.3	0.869
0.930	4	0.4	1.181
0.978	5	0.5	1.517
0.994	6	0.6	1.890
0.999	7	0.7	2.321
1.000	8	0.8	2.859
1.000	9	0.9	3.645

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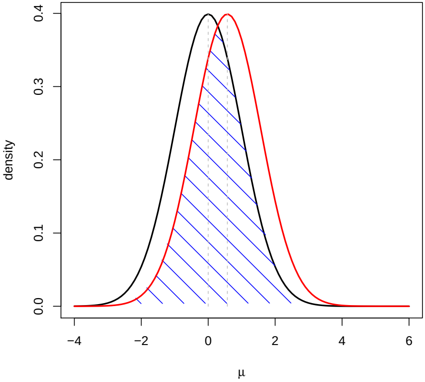
Example Calibration

Densities: N(0,1) vs N(0.571,1) with $h = 0.02$



Hellinger distance for marginal posterior distributions

– INLA provides approximations of the marginal posterior distributions of the latent Gaussian field.

– `inla.hyperpar()`

– Function `inla.smarginal()` allows the evaluation and extrapolation of the posteriors at a fixed grid of values.

⇒ Numerical integration

$$H(post(\boldsymbol{\theta}_0), post(\boldsymbol{\theta})) \approx \sqrt{1 - \sum_k \sqrt{post(\boldsymbol{\theta}_0)(k) post(\boldsymbol{\theta})(k)} \Delta_k}$$

Sensitivity measure and its calibration

- Sensitivity measure:

  Default $\boldsymbol{\theta}_0$ and a shifted $\boldsymbol{\theta}$ values of the prior

  $pri(\boldsymbol{\theta})$: the prior (no-data) density

  $i(\boldsymbol{\theta}) = H(pri(\boldsymbol{\theta}_0), pri(\boldsymbol{\theta}))$

  $post(\boldsymbol{\theta})$: the corresponding posterior density

  $d(\boldsymbol{\theta}) = H(post(\boldsymbol{\theta}_0), post(\boldsymbol{\theta}))$

$$\boxed{S(\boldsymbol{\theta}_0, \boldsymbol{\theta}) = \frac{d(\boldsymbol{\theta})}{i(\boldsymbol{\theta})}}$$

- Calibration:

$$C(S, 0.5) = \mu(S \times h(0.5))$$

Infinitesimal Sensitivity

$$\boxed{S^*(\boldsymbol{\theta}_0) = \lim_{\boldsymbol{\theta}\to\boldsymbol{\theta}_0} S(\boldsymbol{\theta}_0, \boldsymbol{\theta}) = \lim_{\boldsymbol{\theta}\to\boldsymbol{\theta}_0} \frac{d(\boldsymbol{\theta})}{i(\boldsymbol{\theta})}}$$

McCulloch (1989)

Let $\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \boldsymbol{\epsilon}$ and apply the Taylor expansion to obtain

$$d(\boldsymbol{\theta}) \approx \boldsymbol{\epsilon}^T D^2 d(\boldsymbol{\theta}_0)\boldsymbol{\epsilon}/2, \qquad i(\boldsymbol{\theta}) \approx \boldsymbol{\epsilon}^T D^2 i(\boldsymbol{\theta}_0)\boldsymbol{\epsilon}/2$$

and

$$S(\boldsymbol{\theta}_0, \boldsymbol{\theta}_0 + \boldsymbol{\epsilon}) \approx \boldsymbol{\epsilon}^T D^2 d(\boldsymbol{\theta}_0)\boldsymbol{\epsilon}/\boldsymbol{\epsilon}^T D^2 i(\boldsymbol{\theta}_0)\boldsymbol{\epsilon}$$

Infinitesimal Sensitivity

McCulloch (1989) (continued)

Let λ^* be the dominant eigenvalue and $\boldsymbol{\epsilon}^*$ the corresponding eigenvector of

$$(D^2 i(\boldsymbol{\theta}_0))^{-1} D^2 d(\boldsymbol{\theta}_0)$$

Take (Srivastava and Carter 1983)

$$\boxed{S^*(\boldsymbol{\theta}_0) \approx \lambda^*}$$

$\boldsymbol{\epsilon}^*$ indicates which linear combination of the elements of $\boldsymbol{\theta}_0$ is most influential.

$D^2 d(\boldsymbol{\theta}_0)$ and $D^2 i(\boldsymbol{\theta}_0)$ can be approximated numerically.

A Conjugate Example (Box, 1980)

$Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} \mathrm{N}(m, \kappa^{-1})$, where κ is fixed and $m \sim \mathrm{N}(\mu, \lambda^{-1})$ then

$$m \mid \mathbf{y} \sim \mathrm{N}\left(\frac{n\kappa\bar{y} + \lambda\mu}{n\kappa + \lambda}, (n\kappa + \lambda)^{-1}\right)$$

Assume $n = 4$, $\kappa = 1$ and $\bar{y} = 76$

Prior of interest: $\boldsymbol{\theta}_0 = (\mu_0, \lambda_0) = (70, 0.5)$

$S^*(\boldsymbol{\theta}_0) = \lambda^* = 1.79$ and $C(S^*(\boldsymbol{\theta}_0), 0.5) = 0.91$ are mainly caused by λ_0

as $\epsilon^* = (0.113, -0.994)$

Output from `inla`

```
Fixed effects:
const
mean    sd 0.025quant 0.5quant 0.975quant kld
75.33 0.47    74.40    75.33      76.26    0
```

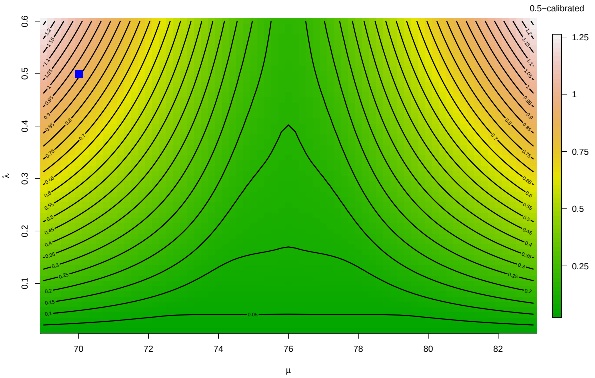
A possible output from `inla`

```
Fixed effects:

const

mean    sd 0.025quant 0.5quant 0.975quant kld sens c.sens

75.33 0.47    74.40    75.33      76.26    0  1.79  0.91

ev.mu ev.lambda

0.113 -0.994
```

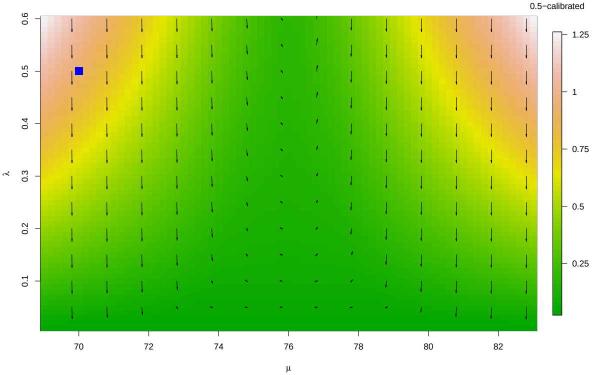
Prior used: $\boldsymbol{\theta}_0 = (\mu_0, \lambda_0) = (70, 0.5)$

Computational time: 3s

0.5-Calibrated Sensitivity Landscape from `inla`



0.5-Calibrated Sensitivity Landscape from `inla` with Eigenvectors



Prior-data conflict

– Box (1980):

Does the prior place most of its mass on parameter values that are not feasible in light of the data?

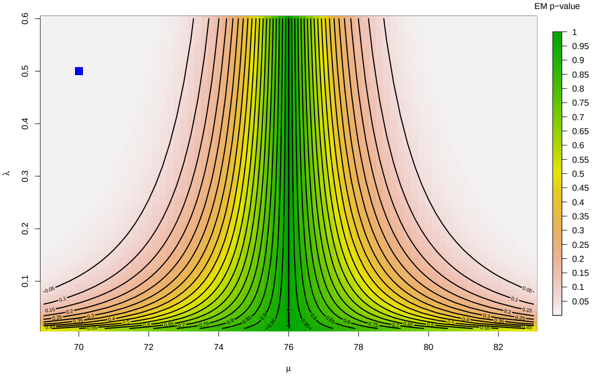
$$\text{p-value} = \mathrm{P}[p(\mathbf{x}) \leq p(\mathbf{x}_{obs})],$$

where $p(\mathbf{x})$ is the prior predictive distribution

– Evans and Moshonov (2006) restricted the Box's approach to minimal sufficient statistics.

Prior-data conflict: Landscape

Box (1980): p = 3.2e-05 and Evans and Moshonov (2006): p = 6.3e-05



Non-conjugate example: Disease Mapping

Fong, Rue, and Wakefield (2010): Lip Cancer in Scotland (1975-1980)

$$Y_j \mid \mu_j \overset{ind}{\sim} \mathrm{Po}(\mu_j),\ j = 1, \ldots, J$$

$$\begin{aligned} \log \mu_j &= \log e_j + c + v_j & (1) \\ \log \mu_j &= \log e_j + c \qquad + u_j & (2) \\ \log \mu_j &= \log e_j + c + v_j + u_j & (3) \end{aligned}$$

const: $c \sim \mathrm{N}(\mu, \lambda^{-1})$

iid: $v_j \overset{iid}{\sim} \mathrm{N}(0, \kappa_v^{-1})$ with $\kappa_v \sim \mathrm{G}(\alpha, \beta)$

ICAR: $u_j \sim \mathrm{ICAR}(\kappa_u^{-1})$ with $\kappa_u \sim \mathrm{G}(\gamma, \delta)$ and `constr = TRUE`

Prior of interest:

$$\boldsymbol{\theta}_0 = (\mu_0, \lambda_0, \alpha_0, \beta_0, \gamma_0, \delta_0) = (\overbrace{0.1, 0.01}^{c}, \overbrace{1, 0.14}^{v}, \overbrace{1, 0.2/0.59}^{u})$$

Computation

– $i(\boldsymbol{\theta}_0)$, $d(\boldsymbol{\theta}_0)$:

Disjoint Support Decomposition (DSD)

$$BC(f_0 g_0 h_0, f_1 g_1 h_1) \stackrel{DSD}{=} BC(f_0, f_1) BC(g_0, g_1) BC(h_0, h_1)$$

– $d(\boldsymbol{\theta}_0)$: number of `inla` calls needed

$d = 4$ for "const + iid" and "const + ICAR" $\Rightarrow$ n.calls $= 33$

$d = 6$ for "const + iid + ICAR" $\Rightarrow$ n.calls $= 73$

In general n.calls $= 1 + 2d^2$

Results

Prior of interest:

$$\boldsymbol{\theta}_0 = (\mu_0, \lambda_0, \alpha_0, \beta_0, \gamma_0, \delta_0) = (\overbrace{0.1, 0.01}^{c}, \overbrace{1, 0.14}^{v}, \overbrace{1, 0.2/0.59}^{u})$$

iid	ICAR	$S^*(\theta_0) = \lambda^*$	ϵ^*						$C(S^*, 0.5)$
			$(\mu_0,$	$\lambda_0,$	$\alpha_0,$	$\beta_0,$	$\gamma_0,$	$\delta_0)$	
yes 31s	no	0.199	“0”	“0”	“1”	“0”			0.099
no 65s	yes	0.240	“0”	“0”			“-1”	“0”	0.119
yes 180s	yes	1.304	“0”	“0”	“0”	“-1”	“0”	“0”	0.656

Results (continued)

Sensitivities for each parameter can be obtained by a shortcut:

Look at

$$(D^2 i(\boldsymbol{\theta}_0))^{-1} D^2 d(\boldsymbol{\theta}_0),$$

where $D^2 i$ and $D^2 d$ are diagonal matrices i.e. no mixed second partial derivatives have to be computed!

iid	ICAR	$(\mu_0,$	$\lambda_0,$	$\alpha_0,$	$\beta_0,$	$\gamma_0,$	$\delta_0)$
yes	no	0.012	1e-04	0.198	0.065		
no	yes	0.005	5e-05			0.230	0.140
yes	yes	0.619	2e-04	0.403	1.284	0.274	0.237

Conclusions

- Calibration with respect to the normal distribution: $h \in (0, 1)$
- Relationship to prior-data conflict measures
- Perfect approximation of the analytical posterior distribution provided by `inla`
- An improvement of a few computational issues is necessary
- Approximate computation of infinitesimal sensitivity with `inla` is possible
- Non-conjugate models can be tackled

Thank you for your attention!

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