Bayesian Adaptive Smoothing Spline Using Stochastic Differential Equations (SDEs)

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Outline

- Background and motivation
- SDEs for adaptive smoothing
- Modeling of local smoothing function
- MCMC and INLA algorithms
- Simulated example
- Discussion

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Classic smoothing spline

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- Nonparametric regression:

$$y_i = f(t_i) + \varepsilon_i, \quad \varepsilon_i \overset{iid}{\sim} N\left(0, \tau^{-1}\right)$$

for $i = 1, \ldots, n$, where $t_i \in \mathcal{T} \subset \mathbf{R}^1$ and f is unknown but smooth.

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- $\widehat{f}$ is smoothing spline (e.g. Wahba, 1990) if

$$\widehat{f} = \arg\min_f \left[\sum_{i=1}^n \left(y_i - f(t_i) \right)^2 + \lambda \int \left(f^{(p)}(t) \right)^2 dt \right]$$

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- Choose λ by e.g., CV or GCV.

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- Hierarchical model:

$$\begin{aligned} \mathbf{y} &= \mathbf{T}\boldsymbol{\beta} + \mathbf{z} + \varepsilon, \quad \varepsilon \sim N\left(\mathbf{0}, \tau^{-1}\mathbf{I}\right) \\ \boldsymbol{\beta} &\propto 1, \quad \mathbf{z} \sim N\left(\mathbf{0}, \delta^{-1}\boldsymbol{\Sigma}\right), \quad \delta = \lambda\tau \end{aligned}$$

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- Negative log posterior:

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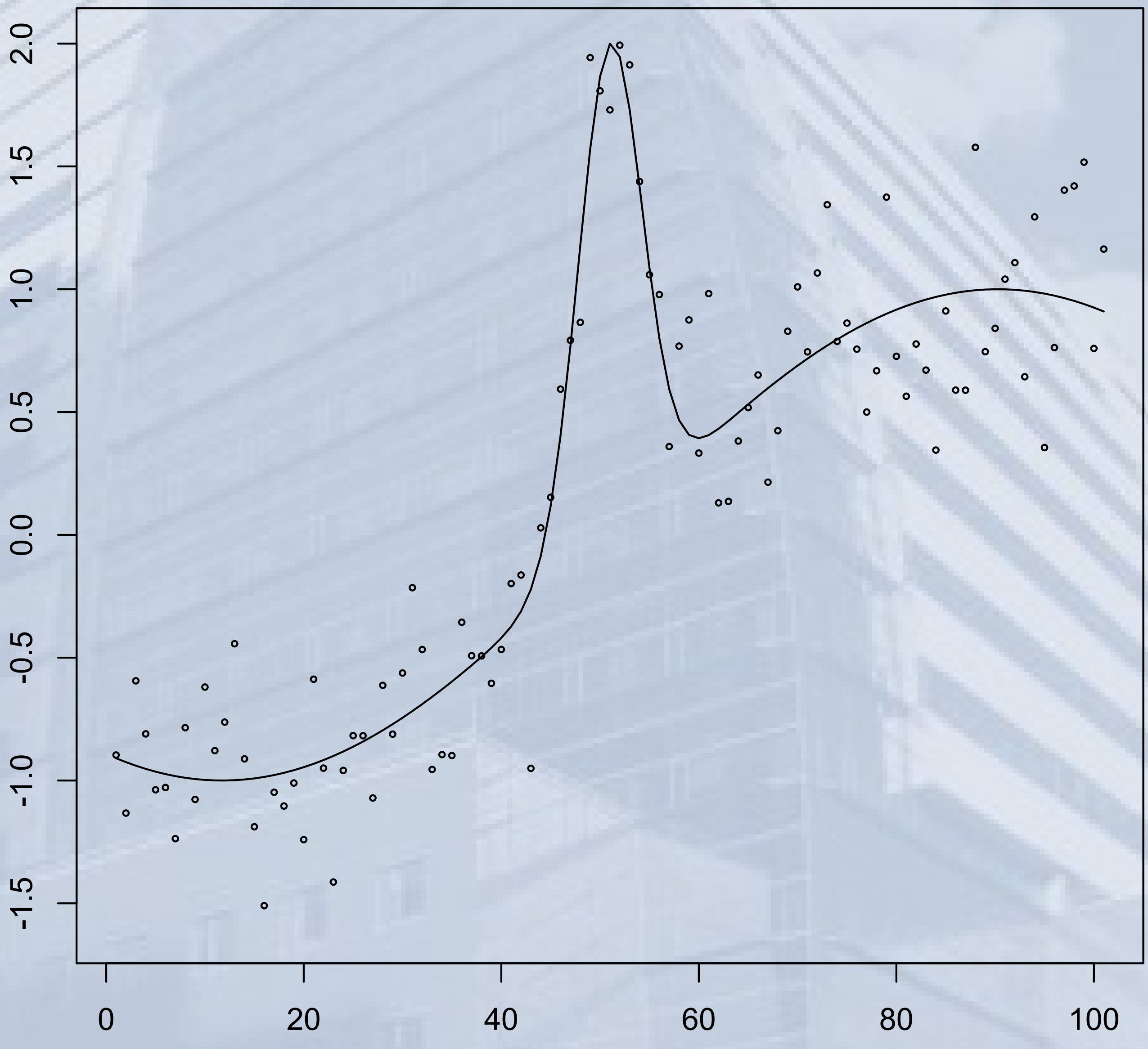
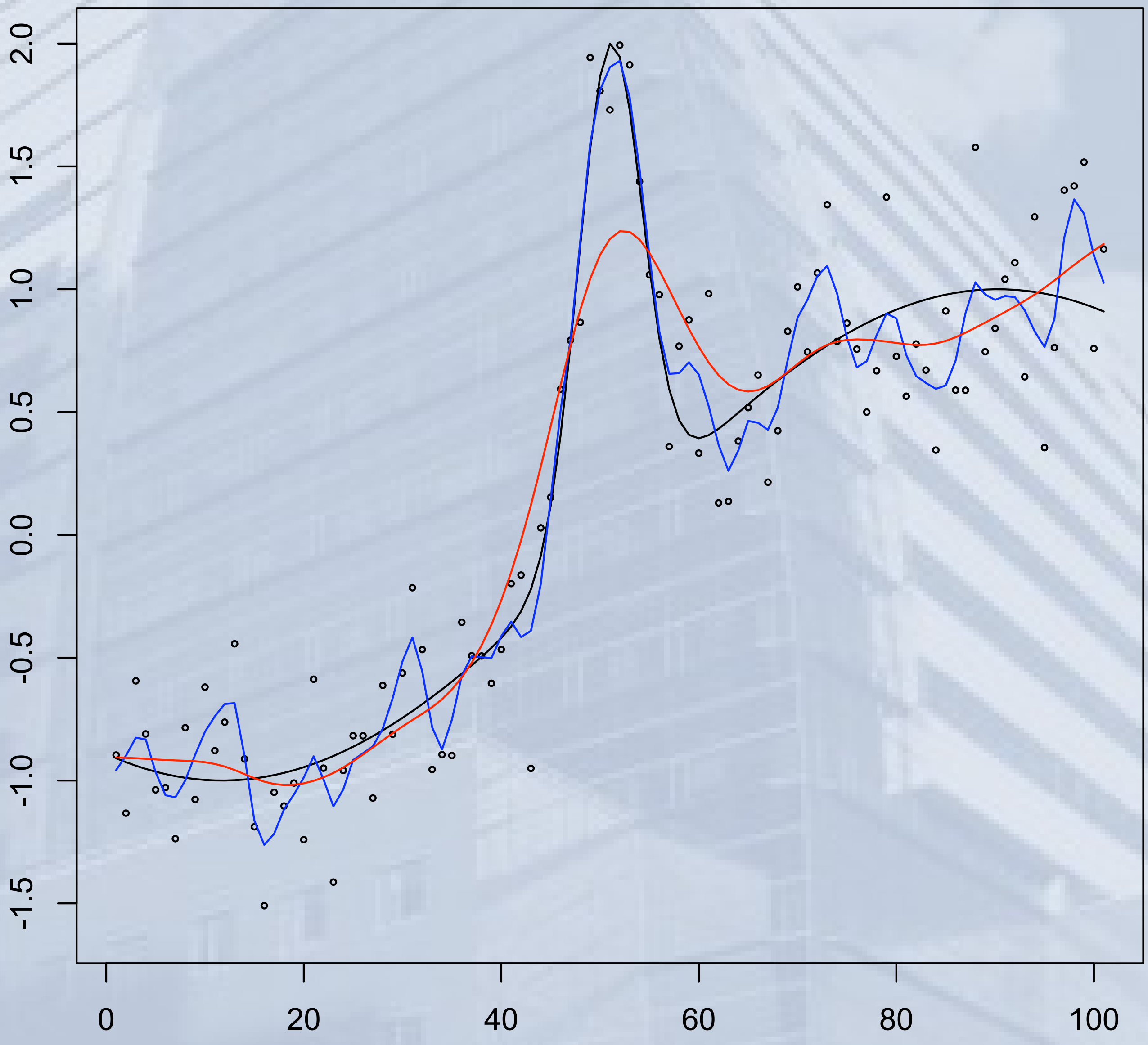
- Straightforward MCMC, but inefficient for big n.

Inhomogeneous function



Problem: single smoothing parameter



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- Hybrid adaptive splines (Luo & Wahba, 1997)
- Local generalized cross-validation (Cummins et al., 2001)
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P-splines

- Model smoothing parameters with another P-spline (e.g., Lang & Brezger, 2004; Baladandayuthapani et al., 2005; Crainiceanu et al., 2007; Krivobokova et al., 2008; Scheipl & Kneib, 2009)

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- Let $t_1 < t_2 < \cdots < t_n$ be fixed points.
- Approximate f as

$$f(t) \approx \sum_{i=1}^{n} \psi_i(t) w_i$$

  – ψ_i: basis functions.

  – w_i: random weights.

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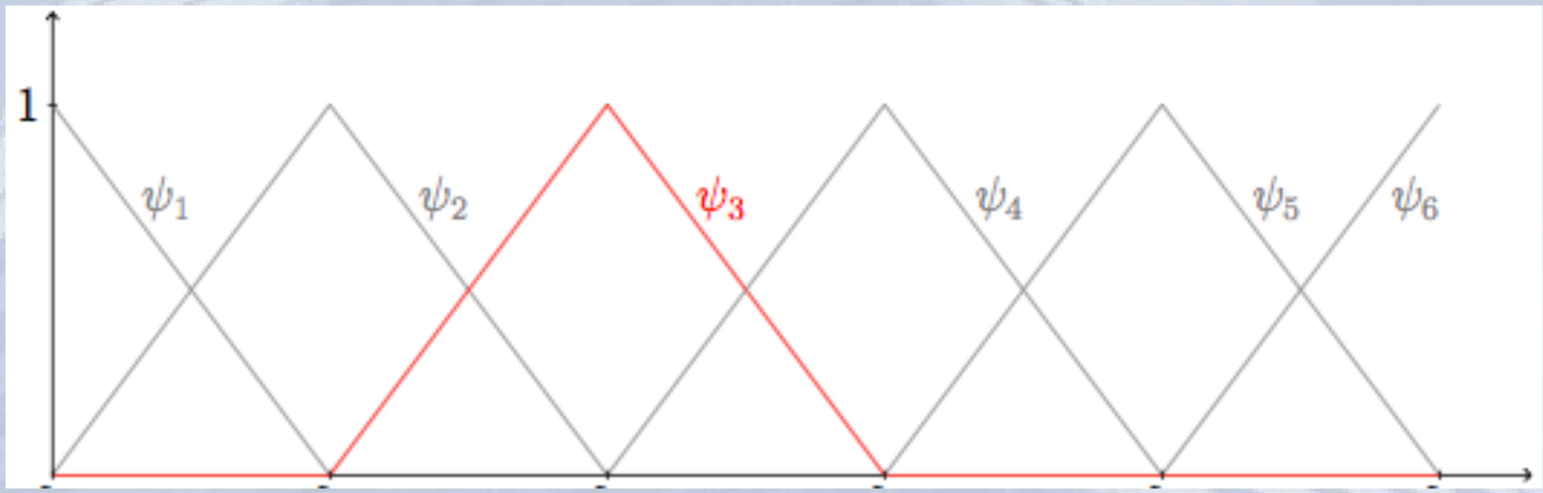
Piecewise linear basis



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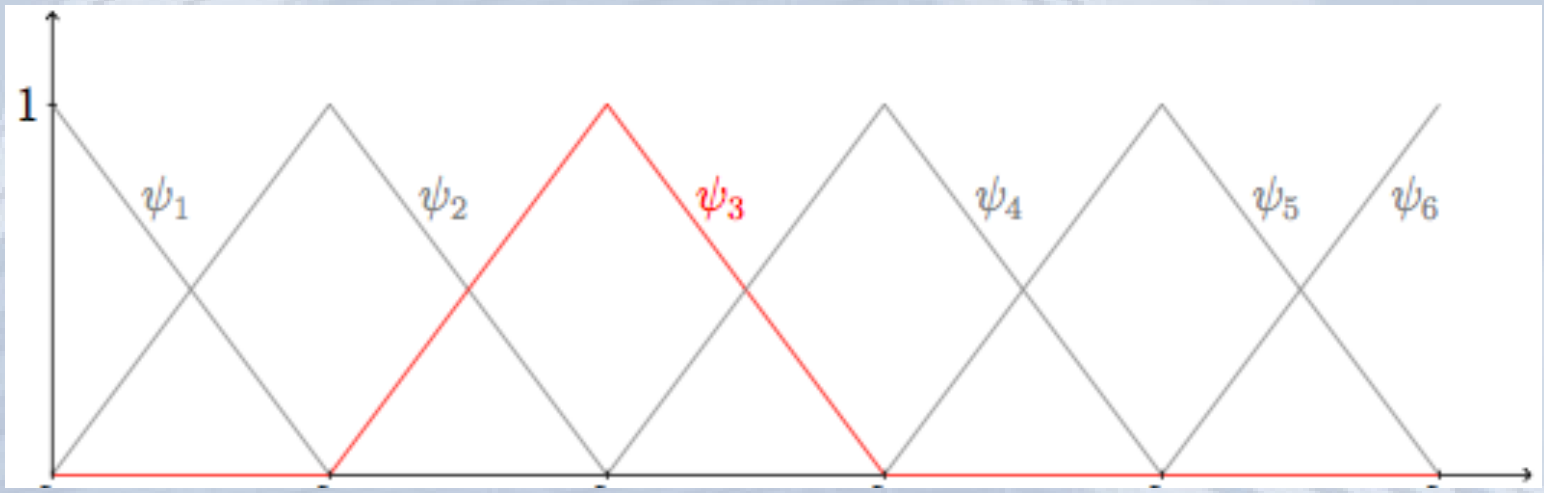

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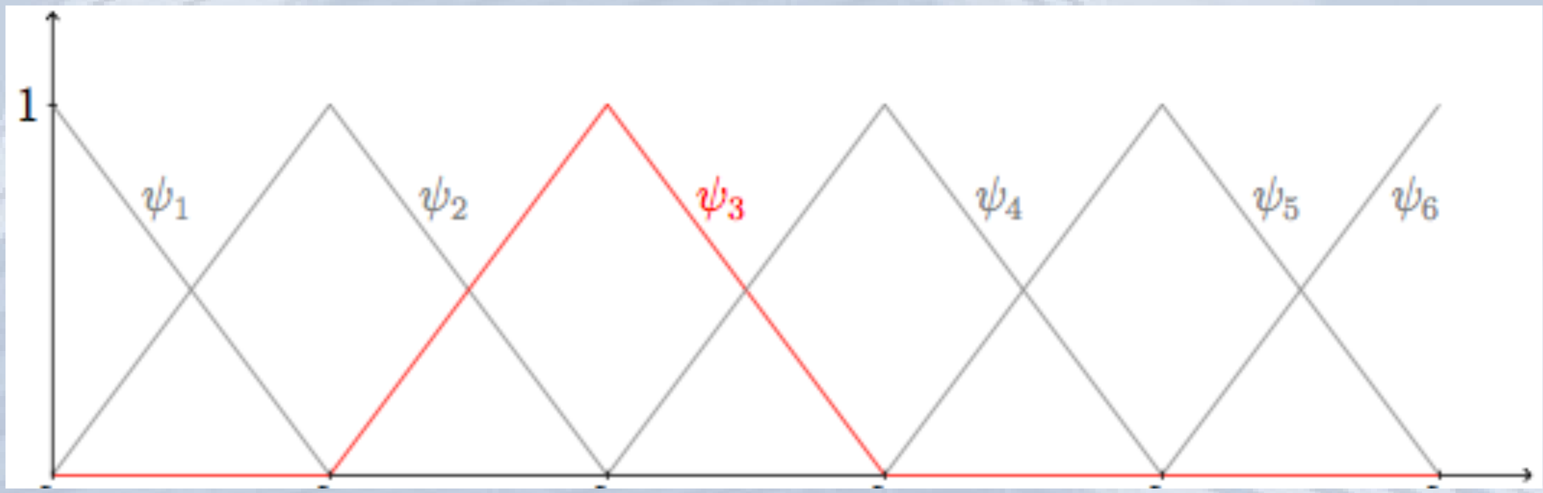

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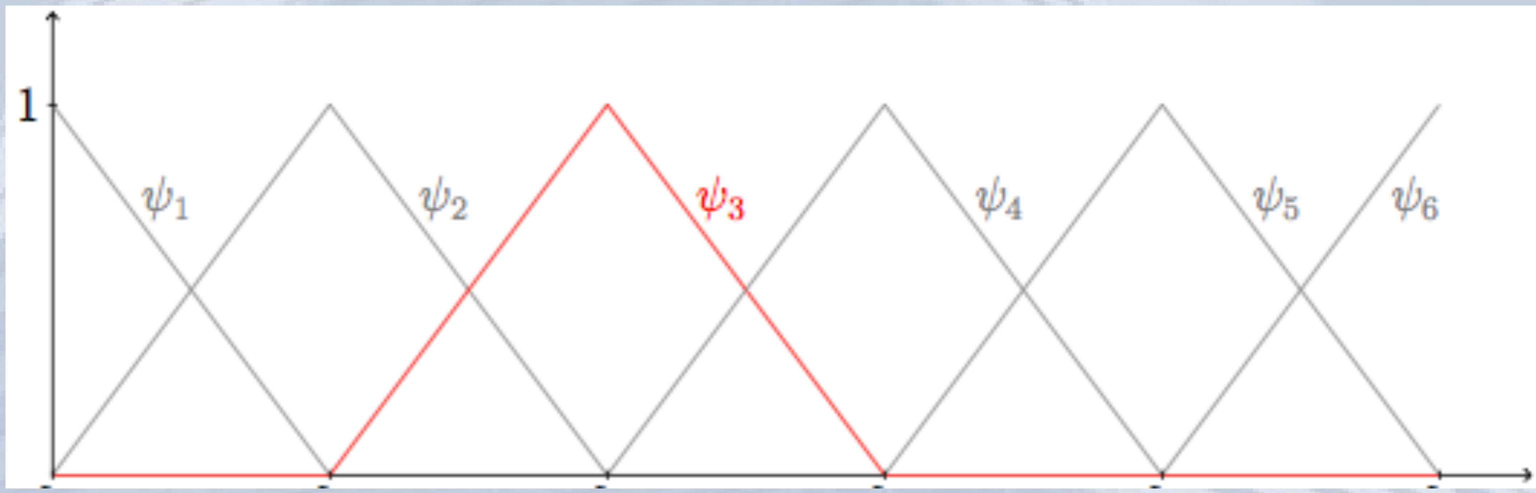

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- Little gain with more complicated basis functions.
- Nice theoretical results are available.

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$$\sum_{i=1}^{n} w_i \left\langle \psi_j, d^2\psi_i/dt^2 \right\rangle \stackrel{d}{=} \left\langle \psi_j, dW/dt \right\rangle, \quad j = 1, \ldots, n. \tag{3}$$

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- LHS of (3) = $\mathbf{Hw}$, where

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- Easy extension to adaptive smoothing.

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  - Closed-form solution by reproducing kernels in Hilbert space.

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Adaptive SDE v1

- Formulation I

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- Possible models for $\nu(t)$: P-splines, Fourier representation, etc.

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- Other parameters can be updated easily.

INLA for adaptive SDE v2

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- Caution: expert's use only!

```
spde <- inla.spde2.generic(M0, M1, M2, B0, B1, B2,
         theta.mu, theta.Q, transform="identity", BLC = B0)
data <- list(y=y, x=x)
formula <- y~ -1 + f(x, model=spde)
result <- inla(formula, data=data, family=‘normal’)
fhat <- result$summary.random$x$mean
```

Simulation study

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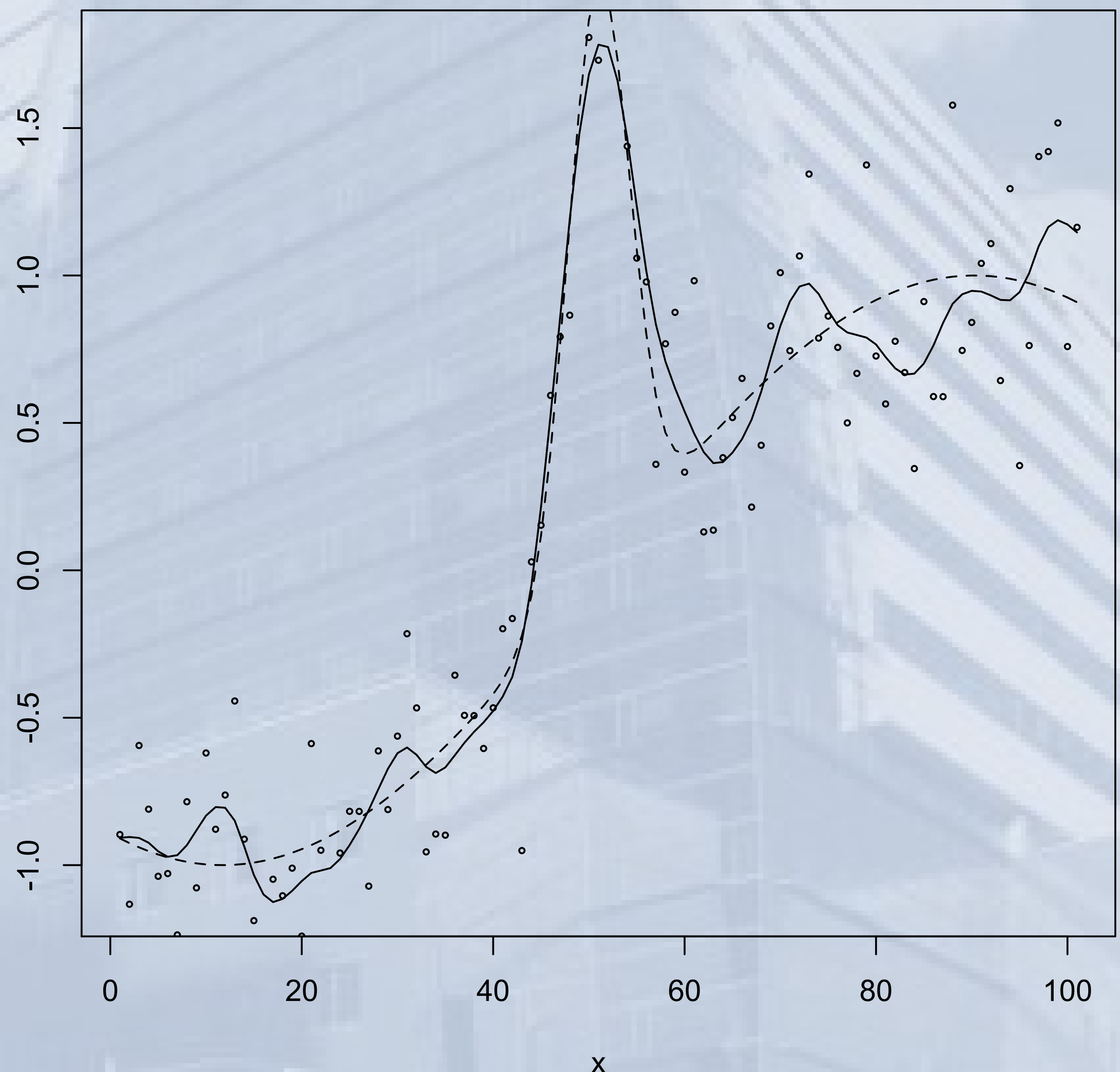
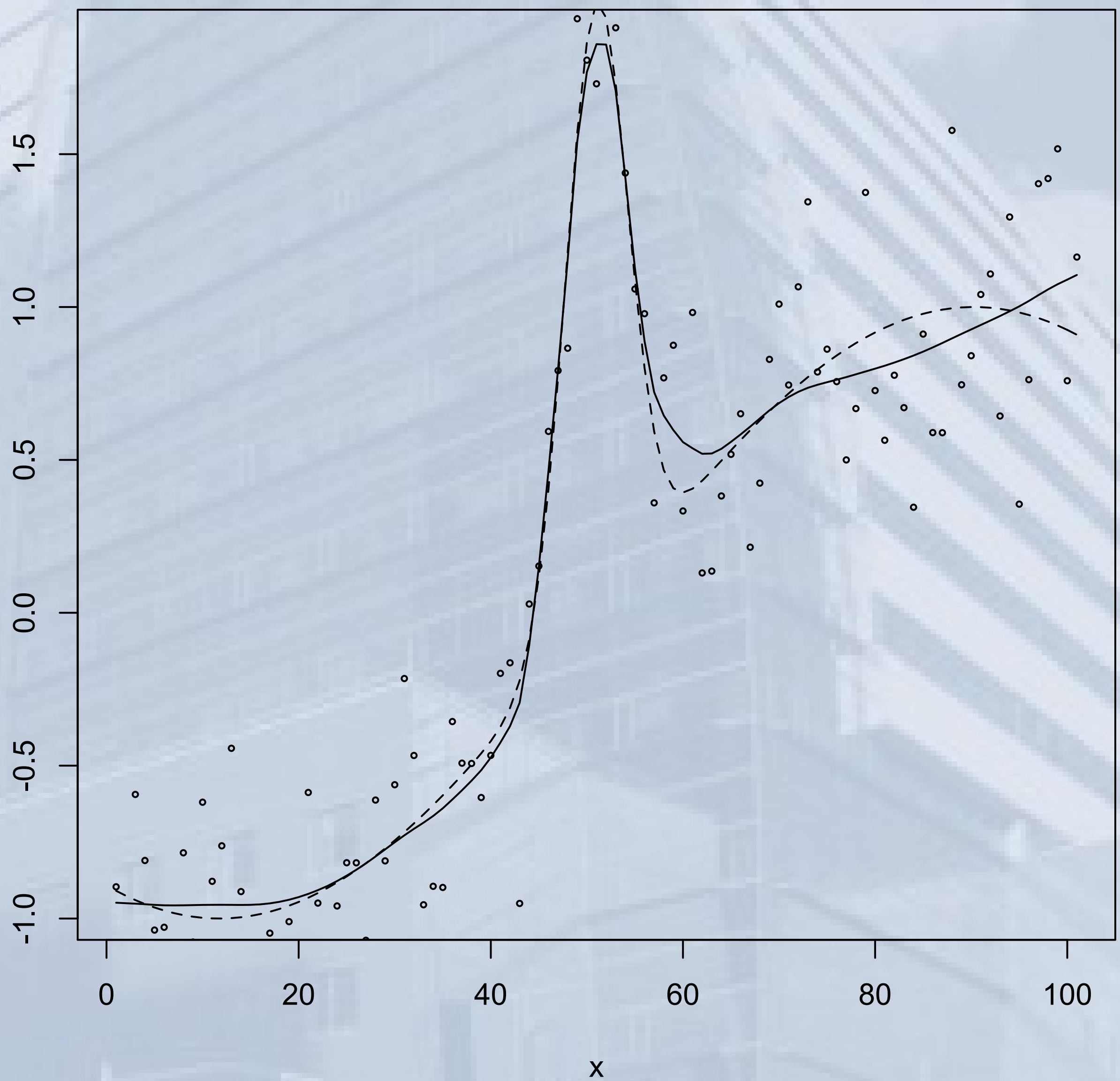
- Ordinary smoothing spline (use `smooth.spline()` in R)



Simulation study

- BASS SDE v1 (use MCMC)



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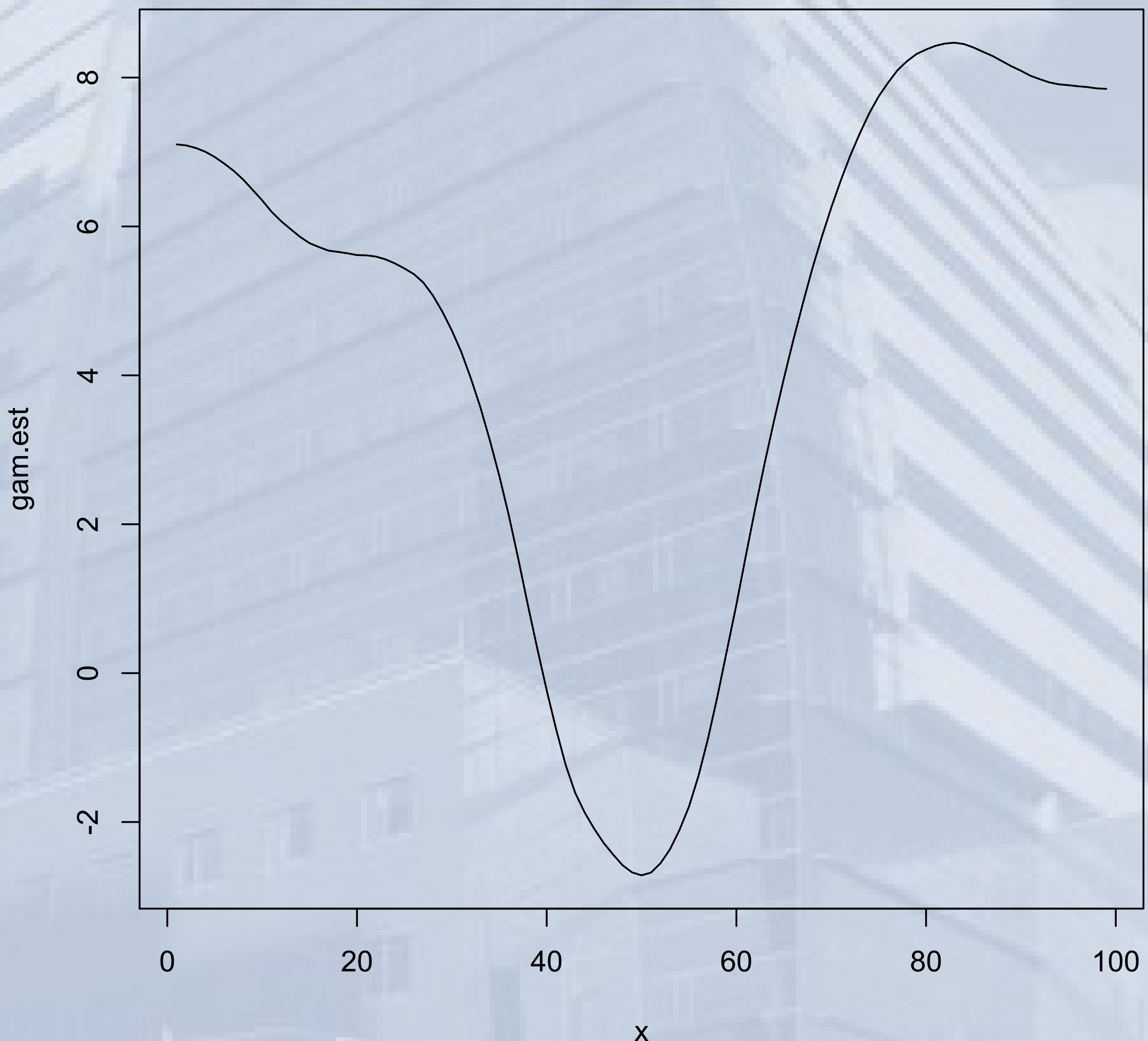
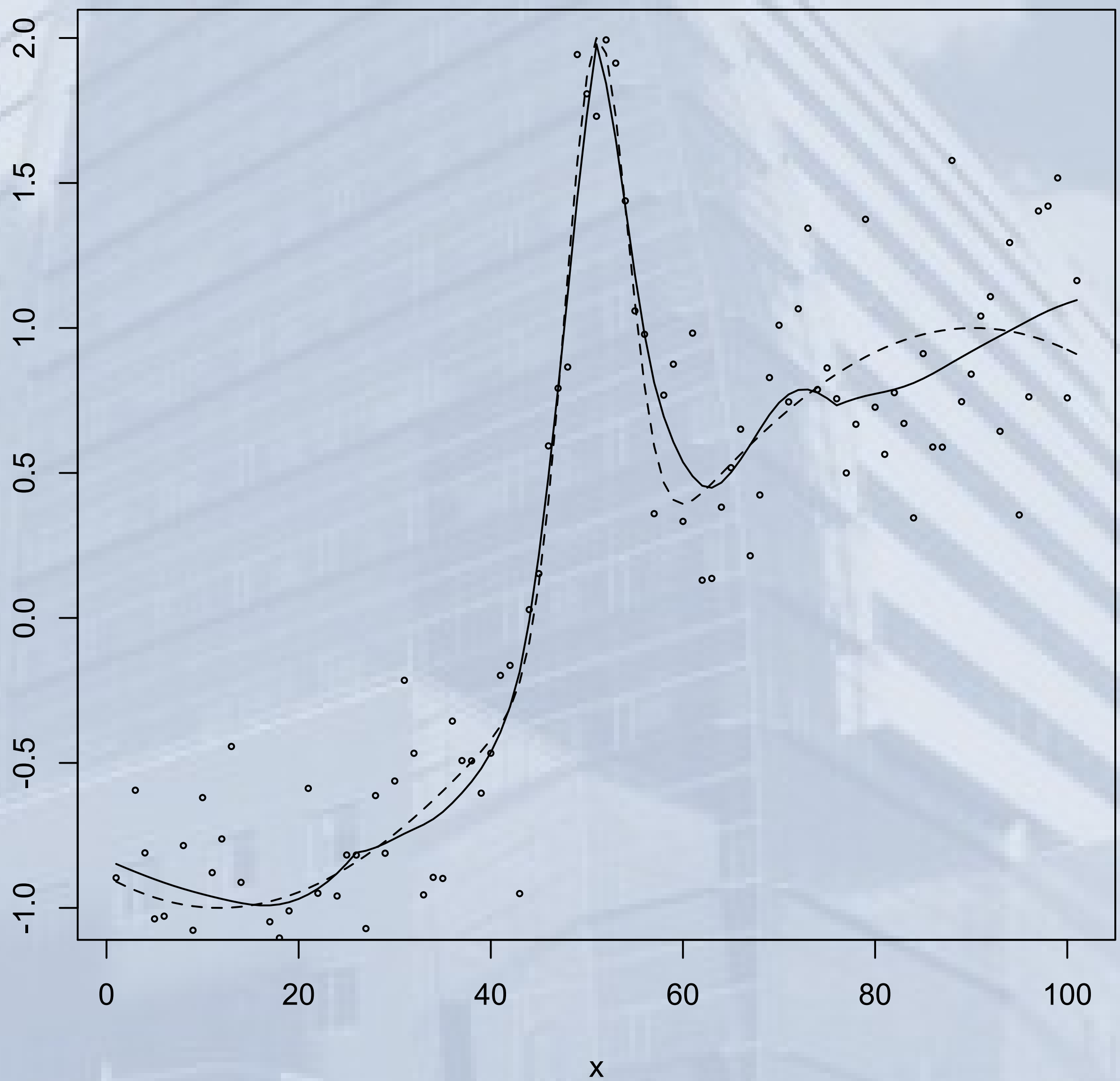


Simulation study

- BASS SDE v2 (use INLA)



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- A better MCMC algorithm?
- Two dimensional extension: thin-plate splines.

Key references

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